OPERATIONS RESEARCH
DOI 10.1287/opre.1080.0571ec
pp. ec1–ec10

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Electronic Companion—"Controlling Coepidemics: Analysis of HIV and Tuberculosis Infection Dynamics" by Elisa F. Long, Naveen K. Vaidya, and Margaret L. Brandeau, *Operations Research*, DOI 10.1287/opre.1080.0571.

Controlling Co-epidemics: Analysis of HIV and Tuberculosis Infection Dynamics

Technical Appendix

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Appendix A: Analytical Details and Proofs of Propositions

A.1. SI × SI Model Basic Reproduction Number

We first compute the new infections matrix, $\mathcal{F}$, and the transfer matrix, $\mathcal{V}$, according to the formula:

$$\left[\mathcal{F}-\mathcal{V}\right]=\begin{bmatrix}\frac{\partial(dI_1/dt)}{\partial I_1} & \frac{\partial(dI_1/dt)}{\partial I_2} & \frac{\partial(dI_1/dt)}{\partial I_3}\\ \frac{\partial(dI_2/dt)}{\partial I_1} & \frac{\partial(dI_2/dt)}{\partial I_2} & \frac{\partial(dI_2/dt)}{\partial I_3}\\ \frac{\partial(dI_3/dt)}{\partial I_1} & \frac{\partial(dI_3/dt)}{\partial I_2} & \frac{\partial(dI_3/dt)}{\partial I_3}\end{bmatrix}. \tag{A.1}$$

To calculate $\mathcal{F}$ and $\mathcal{V}$, we only consider equations (2-4), which correspond to the infected groups (I_1, I_2, I_3) capable of transmitting at least one disease. The non-negative matrix, $\mathcal{F}$, corresponding to new infections in the population is:

$$\mathcal{F}=\begin{bmatrix}u^T\\ v^T\\ w^T\end{bmatrix}, \tag{A.2}$$

where

$$u=\begin{bmatrix}-\frac{\beta S(I_1+I_3)}{N^2}+\frac{\beta S}{N}-\frac{\tau(I_2+I_3)}{N}+\frac{\tau I_1(I_2+I_3)}{N^2}\\ -\frac{\beta S(I_1+I_3)}{N^2}+\frac{\tau I_1(I_2+I_3)}{N^2}-\frac{\tau I_1}{N}\\ -\frac{\beta S(I_1+I_3)}{N^2}+\frac{\beta S}{N}+\frac{\tau I_1(I_2+I_3)}{N^2}-\frac{\tau I_1}{N}\end{bmatrix},$$

$$v=\begin{bmatrix} -\frac{\tau S(I_2+I_3)}{N^2}+\frac{\beta I_2(I_1+I_3)}{N^2}-\frac{\beta I_2}{N} \\ -\frac{\tau S(I_2+I_3)}{N^2}+\frac{\tau S}{N}-\frac{\beta(I_1+I_3)}{N}+\frac{\beta I_2(I_1+I_3)}{N^2} \\ -\frac{\tau S(I_2+I_3)}{N^2}+\frac{\tau S}{N}+\frac{\beta I_2(I_1+I_3)}{N^2}-\frac{\beta I_2}{N} \end{bmatrix},$$

$$w=\begin{bmatrix} -\frac{\beta I_2(I_1+I_3)}{N^2}+\frac{\beta I_2}{N}+\frac{\tau(I_2+I_3)}{N}-\frac{\tau I_1(I_2+I_3)}{N^2} \\ \frac{\beta(I_1+I_3)}{N}-\frac{\beta I_2(I_1+I_3)}{N^2}-\frac{\tau I_1(I_2+I_3)}{N^2}+\frac{\tau I_1}{N} \\ -\frac{\beta I_2(I_1+I_3)}{N^2}+\frac{\beta I_2}{N}-\frac{\tau I_1(I_2+I_3)}{N^2}+\frac{\tau I_1}{N} \end{bmatrix},$$

and where ^{T} represents the transpose of the corresponding vector.

The new infections matrix, $\mathcal{F}$, evaluated at the disease-free equlibrium, E_0, is:

$$\mathcal{F}=\begin{bmatrix} \beta & 0 & \beta \\ 0 & \tau & \tau \\ 0 & 0 & 0 \end{bmatrix}. \tag{A.3}$$

The non-singular matrix, $\mathcal{V}$, corresponding to the transfer of individuals into and out of compartments, is:

$$\mathcal{V}=\begin{bmatrix} \mu+\mu_1 & 0 & 0 \\ 0 & \mu+\mu_2 & 0 \\ 0 & 0 & \mu+\mu_1+\mu_2 \end{bmatrix}. \tag{A.4}$$

We then evaluate $\mathcal{F}\mathcal{V}^{-1}$ at the disease-free equilibrium:

$$\mathcal{F}\mathcal{V}^{-1}=\begin{bmatrix} \frac{\beta}{\mu+\mu_1} & 0 & \frac{\beta}{\mu+\mu_1+\mu_2} \\ 0 & \frac{\tau}{\mu+\mu_2} & \frac{\tau}{\mu+\mu_1+\mu_2} \\ 0 & 0 & 0 \end{bmatrix}. \tag{A.5}$$

Finally, we obtain the basic reproduction number, $\Re_0$, as the dominant eigenvalue of the matrix $\mathcal{F}\mathcal{V}^{-1}$:

$$\begin{aligned} \Re_0 &= \rho\left(\mathcal{F}\mathcal{V}^{-1}\right) \\ &= \max\{\Re_0^1, \Re_0^2\} \end{aligned} \tag{A.6}$$

where

$$\Re_0^1=\frac{\beta}{\mu+\mu_1}, \quad \Re_0^2=\frac{\tau}{\mu+\mu_2}.$$

A.2. SI × SI Model Equilibria Analysis

A.2.1. Disease-Free Equilibrium (DFE)

Proof of Proposition 2. *(Global Stability of DFE in SI × SI Model)*
Let $L=(\mu/2)(S-\Lambda/\mu)^2+\Lambda(I_1+I_2+I_3)$ be a Lyapunov function. L is continuous, with $L>0$ for all points other than the disease-free equilibrium E_0, and $L=0$ only at E_0.

$$\frac{dL}{dt}=\mu\left(S-\frac{\Lambda}{\mu}\right)\frac{dS}{dt}+\Lambda\frac{dI_1}{dt}+\Lambda\frac{dI_2}{dt}+\Lambda\frac{dI_3}{dt}$$

$$
\begin{aligned}
&= \mu\left(S - \frac{\Lambda}{\mu}\right)\left[\Lambda - \beta(I_1+I_3)\frac{S}{N} - \tau(I_2+I_3)\frac{S}{N} - \mu S\right] \\
&\quad + \Lambda\beta(I_1+I_3)\frac{S}{N} + \Lambda\tau(I_2+I_3)\frac{S}{N} - \Lambda(\mu+\mu_1)I_1 - \Lambda(\mu+\mu_2)I_2 - \Lambda(\mu+\mu_1+\mu_2)I_3 \\
&= -\mu^2\left(S - \frac{\Lambda}{\mu}\right)^2 - \beta\mu(I_1+I_3)\frac{S^2}{N} - \tau\mu(I_2+I_3)\frac{S^2}{N} + \Phi,
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi &= 2\Lambda\beta(I_1+I_3)\frac{S}{N} + 2\Lambda\tau(I_2+I_3)\frac{S}{N} - \Lambda(\mu+\mu_1)I_1 - \Lambda(\mu+\mu_2)I_2 - \Lambda(\mu+\mu_1+\mu_2)I_3 \\
&\le 2\Lambda\beta(I_1+I_3) + 2\Lambda\tau(I_2+I_3) - \Lambda(\mu+\mu_1)I_1 - \Lambda(\mu+\mu_2)I_2 - \Lambda(\mu+\mu_1+\mu_2)I_3 \\
&= \Lambda(\mu+\mu_1)\left[\frac{2\beta}{\mu+\mu_1} - 1\right]I_1 + \Lambda(\mu+\mu_2)\left[\frac{2\tau}{\mu+\mu_2} - 1\right]I_2 \\
&\quad + \Lambda(\mu+\mu_1+\mu_2)\left[\frac{2(\beta+\tau)}{\mu+\mu_1+\mu_2} - 1\right]I_3 \\
&< 0,
\end{aligned}
$$

by using the hypothesis (7). Therefore, $dL/dt < 0$ everywhere except at E_0, where $dL/dt = 0$. This implies that the disease-free equilibrium E_0 is globally asymptotically stable. □

A.2.2. Quasi-Disease-Free Equilibria (QDFE)

First QDFE. By setting the system of equations (1-4) equal to 0 and substituting $I_2 = I_3 = 0$, we find the critical points of the system:

$$
\begin{aligned}
\Lambda - \beta I_1\frac{S}{N} - \mu S &= 0, \\
\left(\beta\frac{S}{N} - \mu - \mu_1\right)I_1 &= 0.
\end{aligned} \tag{A.7}
$$

Equation (A.7) gives $I_1 = 0$ (corresponding to the DFE) or

$$
\beta\frac{S}{N} - \mu - \mu_1 = 0. \tag{A.8}
$$

Solving equations (A.7), (A.8) and using $N = S + I_1$, we calculate the first QDFE to be:

$$
E_1 = \left(\frac{\Lambda}{\beta-\mu_1}, \frac{\Lambda(\Re_0^1 - 1)}{\beta-\mu_1}, 0, 0\right). \tag{A.9}
$$

E_1 exists if $\Re_0^1 > 1$.

We calculate the local stability of the first QDFE, E_1, using the following method. We calculate the Jacobian, J, of the system (1-4) which is a matrix of partial derivatives with respect to the variables S, I_1, I_2, I_3. We then evaluate the Jacobian at E_1.

$$
J = \begin{bmatrix} A_{2\times2} & B_{2\times2} \\ 0_{2\times2} & C_{2\times2} \end{bmatrix}, \tag{A.10}
$$

where $0_{2\times 2}$ is zero matrix of order 2×2 and block matrices $A_{2\times 2}$, $B_{2\times 2}$ and $C_{2\times 2}$ are:

$$A = \begin{bmatrix} -\frac{\beta I_1}{N} + \frac{\beta S I_1}{N^2} - \mu & -\frac{\beta S}{N} + \frac{\beta S I_1}{N^2} \\ \frac{\beta I_1}{N} - \frac{\beta S I_1}{N^2} & \frac{\beta S}{N} - \frac{\beta S I_1}{N^2} - \mu - \mu_1 \end{bmatrix},$$

$$B = \begin{bmatrix} -\frac{\tau S}{N} + \frac{\beta S I_1}{N^2} & -\frac{(\tau+\beta)S}{N} + \frac{\beta S I_1}{N^2} \\ -\frac{\tau I_1}{N} - \frac{\beta S I_1}{N^2} & \frac{\beta S}{N} - \frac{\tau I_1}{N} - \frac{\beta S I_1}{N^2} \end{bmatrix},$$

$$C = \begin{bmatrix} -\frac{\beta I_1}{N} + \frac{\tau S}{N} - \mu - \mu_2 & \frac{\tau S}{N} \\ \frac{(\beta+\tau)I_1}{N} & \frac{\tau I_1}{N} - \mu - \mu_1 - \mu_2 \end{bmatrix}.$$

Using equation (A.8), matrix A reduces to:

$$A = \begin{bmatrix} -\frac{\beta I_1}{N} + \frac{\beta S I_1}{N^2} - \mu & -\frac{\beta S}{N} + \frac{\beta S I_1}{N^2} \\ \frac{\beta I_1}{N} - \frac{\beta S I_1}{N^2} & -\frac{\beta S I_1}{N^2} \end{bmatrix}.$$

Since the lower left block is a zero matrix, the eigenvalues of J can be found by calculating the eigenvalues of the block matrices A and C. For $\Re_0^1 > 1$, which is required for existence of E_1, the trace of A is negative and the determinant of A is positive, so all eigenvalues of A are negative. Moreover, the trace of C is negative and the determinant of C is positive, so all eigenvalues of C are negative under the following condition:

$$\mu + \mu_1 + \mu_2 > \tau \left[1 + \frac{\mu_1}{\Re_0^1(\beta - \mu_1 + \mu_2)} \right]. \qquad \text{(A.11)}$$

Proof of Proposition 3. *(Local Stability of First QDFE in SI × SI Model)*
We begin by showing $trace(A) < 0$ and $det(A) > 0$.

$$\begin{aligned} trace(A) &= -\frac{\beta I_1}{N} - \mu < 0. \\ det(A) &= \left(-\frac{\beta I_1}{N} + \frac{\beta S I_1}{N^2} - \mu\right)\left(-\frac{\beta S I_1}{N^2}\right) - \left(-\frac{\beta S}{N} + \frac{\beta S I_1}{N^2}\right)\left(\frac{\beta I_1}{N} - \frac{\beta S I_1}{N^2}\right) \\ &= \frac{\beta S I_1}{N^2}\left[\mu + \beta - \frac{\beta S}{N}\right] \\ &= \frac{\beta S I_1}{N^2}\left[\mu + \beta\left(1 - \frac{1}{\Re_0^1}\right)\right] \qquad \left[\because \frac{S}{N} = \frac{\mu+\mu_1}{\beta}\right] \\ &> 0. \qquad [\because \Re_0^1 > 1] \end{aligned}$$

Therefore all eigenvalues of A are negative.

We now determine the condition under which $trace(C) < 0$ and $det(C) > 0$.

$$\begin{aligned} trace(C) &= -\frac{\beta I_1}{N} + \frac{\tau S}{N} - \mu - \mu_2 + \frac{\tau I_1}{N} - \mu - \mu_1 - \mu_2 \\ &= \tau - \frac{\beta I_1}{N} - \mu - \mu_2 - \mu - \mu_1 - \mu_2 \qquad [\because S + I_1 = N] \\ &< 0, \end{aligned}$$

if

$$\tau - \left(\frac{\beta I_1}{N} + \mu + \mu_2\right) < \mu + \mu_1 + \mu_2, \tag{A.12}$$

which follows from $det(C) > 0$.

$$\begin{aligned}
det(C) &= \left(-\frac{\beta I_1}{N} + \frac{\tau S}{N} - \mu - \mu_2\right)\left(\frac{\tau I_1}{N} - \mu - \mu_1 - \mu_2\right) - \left(\frac{\tau S}{N}\right)\left(\frac{(\beta+\tau)I_1}{N}\right) \\
&= \left(\frac{\beta I_1}{N} + \mu + \mu_2\right)(\mu + \mu_1 + \mu_2) - \left(\frac{\tau I_1}{N}\right)\left(\frac{\beta I_1}{N} + \mu + \mu_2\right) - \left(\frac{\tau S}{N}\right)\left(\frac{\beta I_1}{N} + \mu + \mu_1 + \mu_2\right) \\
&= \left(\frac{\beta I_1}{N} + \mu + \mu_2\right)(\mu + \mu_1 + \mu_2) - \tau\left(\frac{\beta I_1}{N} + \mu + \mu_2\right) - \left(\frac{\tau S}{N}\right)\mu_1 \qquad [\because S + I_1 = N] \\
&> 0,
\end{aligned}$$

if

$$\left(\frac{\beta I_1}{N} + \mu + \mu_2\right)(\mu + \mu_1 + \mu_2) > \tau\left(\frac{\beta I_1}{N} + \mu + \mu_2\right) + \left(\frac{\tau S}{N}\right)\mu_1,$$

which is equivalent to

$$\mu + \mu_1 + \mu_2 > \tau\left[1 + \frac{\mu_1}{\Re_0^1(\beta - \mu_1 + \mu_2)}\right], \tag{A.13}$$

where we used

$$\begin{aligned}
\frac{S}{N} &= \frac{\mu + \mu_1}{\beta}, \\
\frac{I_1}{N} &= \frac{\beta - \mu - \mu_1}{\beta}.
\end{aligned}$$

It is easy to verify that (A.13) implies (A.12).

Therefore, all eigenvalues of C are negative, provided that (9) holds. Thus, the first quasi-disease-free equilibrium, E_1, is locally asymptotically stable. □

Proof of Proposition 4. *(Local Stability of Second QDFE in SI × SI Model)*
The proof of Proposition 4 is similar to Proposition 3, because of symmetry. □

A.3. SII × SEI Model Equilibria Analysis

A.3.1. Disease-Free Equilibrium (DFE)

Proof of Proposition 5. *(Global Stability of DFE in SII × SEI Model)*
Let $L=(\mu/2)(X_{SS}-\Lambda/\mu)^2+\Lambda(X_{SL}+X_{ST}+X_{IS}+X_{IL}+X_{IT}+X_{AS}+X_{AL}+X_{AT})$ be a Lyapunov function. L is continuous, with $L>0$ for all points other than the disease-free equilibrium E_0, and $L=0$ only at E_0.

$$
\begin{aligned}
\frac{dL}{dt} &= \mu\left(X_{SS}-\frac{\Lambda}{\mu}\right)\frac{dX_{SS}}{dt}+\Lambda\left(\frac{dX_{SL}}{dt}+\frac{dX_{ST}}{dt}+\frac{dX_{IS}}{dt}+\frac{dX_{IL}}{dt}+\frac{dX_{IT}}{dt}+\frac{dX_{AS}}{dt}+\frac{dX_{AL}}{dt}+\frac{dX_{AT}}{dt}\right)\\
&= \mu\left(X_{SS}-\frac{\Lambda}{\mu}\right)\Bigg[\Lambda-\beta_I(X_{IS}+X_{IL}+X_{IT})\frac{X_{SS}}{N}-\beta_A(X_{AS}+X_{AL}+X_{AT})\frac{X_{SS}}{N}\\
&\quad -\tau(X_{ST}+X_{IT}+X_{AT})\frac{X_{SS}}{N}-\mu X_{SS}\Bigg]\\
&\quad +\Lambda\beta_I(X_{IS}+X_{IL}+X_{IT})\frac{X_{SS}}{N}+\Lambda\beta_A(X_{AS}+X_{AL}+X_{AT})\frac{X_{SS}}{N}+\Lambda\tau(X_{ST}+X_{IT}+X_{AT})\frac{X_{SS}}{N}\\
&\quad -\Lambda\mu X_{SL}-\Lambda(\mu+\mu_T)X_{ST}-\Lambda\mu X_{IS}-\Lambda\mu X_{IL}-\Lambda(\mu+\mu_T)X_{IT}\\
&\quad -\Lambda(\mu+\mu_A)X_{AS}-\Lambda(\mu+\mu_A)X_{AL}-\Lambda(\mu+\mu_A+\mu_T)X_{AT}\\
&= -\mu^2\left(X_{SS}-\frac{\Lambda}{\mu}\right)^2-\beta_I\mu(X_{IS}+X_{IL}+X_{IT})\frac{X_{SS}^2}{N}-\beta_A\mu(X_{AS}+X_{AL}+X_{AT})\frac{X_{SS}^2}{N}\\
&\quad -\tau\mu(X_{ST}+X_{IT}+X_{AT})\frac{X_{SS}^2}{N}+\Phi,
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi &= 2\Lambda\beta_I(X_{IS}+X_{IL}+X_{IT})\frac{X_{SS}}{N}+2\Lambda\beta_A(X_{AS}+X_{AL}+X_{AT})\frac{X_{SS}}{N}+2\Lambda\tau(X_{ST}+X_{IT}+X_{AT})\frac{X_{SS}}{N}\\
&\quad -\Lambda\mu X_{SL}-\Lambda(\mu+\mu_T)X_{ST}-\Lambda\mu X_{IS}-\Lambda\mu X_{IL}-\Lambda(\mu+\mu_T)X_{IT}\\
&\quad -\Lambda(\mu+\mu_A)X_{AS}-\Lambda(\mu+\mu_A)X_{AL}-\Lambda(\mu+\mu_A+\mu_T)X_{AT}\\
&\le 2\Lambda\beta_I(X_{IS}+X_{IL}+X_{IT})+2\Lambda\beta_A(X_{AS}+X_{AL}+X_{AT})+2\Lambda\tau(X_{ST}+X_{IT}+X_{AT})\\
&\quad -\Lambda\mu X_{SL}-\Lambda(\mu+\mu_T)X_{ST}-\Lambda\mu X_{IS}-\Lambda\mu X_{IL}-\Lambda(\mu+\mu_T)X_{IT}\\
&\quad -\Lambda(\mu+\mu_A)X_{AS}-\Lambda(\mu+\mu_A)X_{AL}-\Lambda(\mu+\mu_A+\mu_T)X_{AT}\\
&= -\Lambda\mu X_{SL}+\left[2\Lambda\tau-\Lambda(\mu+\mu_T)\right]X_{ST}\\
&\quad +\left[2\Lambda\beta_I-\Lambda\mu\right]X_{IS}+\left[2\Lambda\beta_I-\Lambda\mu\right]X_{IL}+\left[2\Lambda(\beta_I+\tau)-\Lambda(\mu+\mu_T)\right]X_{IT}\\
&\quad +\left[2\Lambda\beta_A-\Lambda(\mu+\mu_A)\right]X_{AS}+\left[2\Lambda\beta_A-\Lambda(\mu+\mu_A)\right]X_{AL}+\left[2\Lambda(\beta_A+\tau)-\Lambda(\mu+\mu_A+\mu_T)\right]X_{AT}\\
&= -\Lambda\mu X_{SL}+\Lambda(\mu+\mu_T)\left[\frac{2\tau}{\mu+\mu_T}-1\right]X_{ST}
\end{aligned}
$$

$$
\begin{aligned}
&+\Lambda\mu\left[\frac{2\beta_I}{\mu}-1\right]X_{IS}+\Lambda\mu\left[\frac{2\beta_I}{\mu}-1\right]X_{IL}+\Lambda(\mu+\mu_T)\left[\frac{2(\beta_I+\tau)}{\mu+\mu_T}-1\right]X_{IT}\\
&+\Lambda(\mu+\mu_A)\left[\frac{2\beta_A}{\mu+\mu_A}-1\right]X_{AS}\\
&+\Lambda(\mu+\mu_A)\left[\frac{2\beta_A}{\mu+\mu_A}-1\right]X_{AL}+\Lambda(\mu+\mu_A+\mu_T)\left[\frac{2(\beta_A+\tau)}{\mu+\mu_A+\mu_T}-1\right]X_{AT}\\
&<0,
\end{aligned}
$$

provided that

$$\frac{\tau}{\mu+\mu_T}<\frac{1}{2},\quad \frac{\beta_I}{\mu}<\frac{1}{2},\quad \frac{\beta_I+\tau}{\mu+\mu_T}<\frac{1}{2},\quad \frac{\beta_A}{\mu+\mu_A}<\frac{1}{2},\quad \frac{\beta_A+\tau}{\mu+\mu_A+\mu_T}<\frac{1}{2}.$$

We note that

$$\frac{\beta_I+\tau}{\mu+\mu_T}<\frac{1}{2}$$

implies

$$\frac{\tau}{\mu+\mu_T}<\frac{1}{2}.$$

These inequalities are equivalent to the hypothesis (24). Therefore, $dL/dt<0$ everywhere except at E_0, where $dL/dt=0$. This implies that the disease-free equilibrium E_0 is globally asymptotically stable. □

Appendix B: SI × SI Model with Recovery

We extend the basic SI × SI model to include recovery from disease. In this case, recovery means that an individual is completely cured of a disease and can no longer infect others. We assume that individuals infected with disease 1 recover at rate ρ_1, and individuals infected with disease 2 recover at rate ρ_2. We assume that once an individual recovers from a particular disease, he becomes susceptible to that disease again. The SI × SI model with complete recovery does not apply to incurable diseases, such as HIV. Mathematically, we write this model as:

$$\frac{dS}{dt}=\Lambda+\rho_1 I_1+\rho_2 I_2-\beta(I_1+I_3)\frac{S}{N}-\tau(I_2+I_3)\frac{S}{N}-\mu S, \tag{B.1}$$

$$\frac{dI_1}{dt}=\beta(I_1+I_3)\frac{S}{N}+\rho_2 I_3-\tau(I_2+I_3)\frac{I_1}{N}-(\rho_1+\mu+\mu_1)I_1, \tag{B.2}$$

$$\frac{dI_2}{dt}=\tau(I_2+I_3)\frac{S}{N}+\rho_1 I_3-\beta(I_1+I_3)\frac{I_2}{N}-(\rho_2+\mu+\mu_2)I_2, \tag{B.3}$$

$$\frac{dI_3}{dt}=\beta(I_1+I_3)\frac{I_2}{N}+\tau(I_2+I_3)\frac{I_1}{N}-(\rho_1+\rho_2+\mu+\mu_1+\mu_2)I_3, \tag{B.4}$$

where $N=S+I_1+I_2+I_3$.

We now calculate the basic reproduction number and derive the conditions under which the DFE and QDFE are stable. We distinguish the SI × SI model with recovery by using * notation.

B.1. Basic Reproduction Number

The basic reproduction number, $\Re_0^*$, of the SI × SI model with recovery is

$$\Re_0^* = \max\{\Re_0^{1*}, \Re_0^{2*}\}, \tag{B.5}$$

where

$$\Re_0^{1*} = \frac{\beta}{\rho_1 + \mu + \mu_1}, \quad \Re_0^{2*} = \frac{\tau}{\rho_2 + \mu + \mu_2}.$$

Note that the terms $\Re_0^{1*}$ and $\Re_0^{2*}$ are similar to $\Re_0^1$ and $\Re_0^2$ in the SI × SI model with no recovery, but with the additional recovery term ρ_1 or ρ_2 in the denominator. Once again, $\Re_0^{1*}$ and $\Re_0^{2*}$ coincide with the basic reproduction numbers for the two single-disease models with recovery.

B.2. SI × SI Model with Recovery Equilibria Analysis

B.2.1. Disease-Free Equilibrium (DFE) The disease-free equilibrium for equations (B.1-B.4) is the same as that for the system given by equations (1-4):

$$E_0^* = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right). \tag{B.6}$$

Proposition B1. *(Local Stability of DFE in SI × SI Model with Recovery)*
The disease-free equilibrium E_0^* is locally asymptotically stable if $\Re_0^* < 1$.

Proposition B2. *(Global Stability of DFE in SI × SI Model with Recovery)*
Consider the domain $\Omega = \{(S, I_1, I_2, I_3) : 0 \le S \le \bar{S}, I_1 \ge 0, I_2 \ge 0, I_3 \ge 0\}$ and suppose

$$\max\left\{\frac{\beta + (\mu\bar{\mathrm{S}} - \Lambda)(\frac{\rho_1}{2\Lambda})}{\rho_1 + \mu + \mu_1}, \frac{\tau + (\mu\bar{\mathrm{S}} - \Lambda)(\frac{\rho_2}{2\Lambda})}{\rho_2 + \mu + \mu_2}, \frac{\beta + \tau}{\mu + \mu_1 + \mu_2}\right\} < \frac{1}{2}, \tag{B.7}$$

where $\bar{S}$ is the largest value of S such that (B.7) holds. Then the disease-free equilibrium E_0^* is globally asymptotically stable over the domain Ω.

Proof of Proposition B2. *(Global Stability of DFE in SI × SI Model with Recovery)*
Let $L = (\mu/2)(S - \Lambda/\mu)^2 + \Lambda(I_1 + I_2 + I_3)$ be a Lyapunov function. L is continuous, with $L > 0$ for all points other than the disease-free equilibrium E_0^*, and $L = 0$ only at E_0^*.

$$\begin{aligned}
\frac{dL}{dt} &= \mu\left(S - \frac{\Lambda}{\mu}\right)\frac{dS}{dt} + \Lambda\frac{dI_1}{dt} + \Lambda\frac{dI_2}{dt} + \Lambda\frac{dI_3}{dt} \\
&= \mu\left(S - \frac{\Lambda}{\mu}\right)\left[\Lambda + \rho_1 I_1 + \rho_2 I_2 - \beta(I_1 + I_3)\frac{S}{N} - \tau(I_2 + I_3)\frac{S}{N} - \mu S\right] \\
&\quad + \Lambda\beta(I_1 + I_3)\frac{S}{N} + \Lambda\tau(I_2 + I_3)\frac{S}{N} - \Lambda(\rho_1 + \mu + \mu_1)I_1 - \Lambda(\rho_2 + \mu + \mu_2)I_2 - \Lambda(\mu + \mu_1 + \mu_2)I_3 \\
&= -\mu^2\left(S - \frac{\Lambda}{\mu}\right)^2 - \beta\mu(I_1 + I_3)\frac{S^2}{N} - \tau\mu(I_2 + I_3)\frac{S^2}{N} + \Phi,
\end{aligned}$$

where

$$
\begin{aligned}
\Phi &= 2\Lambda\beta(I_1+I_3)\frac{S}{N} + 2\Lambda\tau(I_2+I_3)\frac{S}{N} + (\mu S-\Lambda)\rho_1 I_1 + (\mu S-\Lambda)\rho_2 I_2 \\
&\quad -\Lambda(\rho_1+\mu+\mu_1)I_1 - \Lambda(\rho_2+\mu+\mu_2)I_2 - \Lambda(\mu+\mu_1+\mu_2)I_3 \\
&\leq 2\Lambda\beta(I_1+I_3) + 2\Lambda\tau(I_2+I_3) + (\mu S-\Lambda)\rho_1 I_1 + (\mu S-\Lambda)\rho_2 I_2 \\
&\quad -\Lambda(\rho_1+\mu+\mu_1)I_1 - \Lambda(\rho_2+\mu+\mu_2)I_2 - \Lambda(\mu+\mu_1+\mu_2)I_3 \\
&= \left[2\Lambda\beta + (\mu S-\Lambda)\rho_1 - \Lambda(\rho_1+\mu+\mu_1)\right] I_1 \\
&\quad + \left[2\Lambda\tau + (\mu S-\Lambda)\rho_2 - \Lambda(\rho_2+\mu+\mu_2)\right] I_2 \\
&\quad + \left[2\Lambda(\beta+\tau) - \Lambda(\mu+\mu_1+\mu_2)\right] I_3 \\
&= \Lambda(\rho_1+\mu+\mu_1)\left[\frac{2\beta+(\mu S-\Lambda)(\frac{\rho_1}{\Lambda})}{\rho_1+\mu+\mu_1} - 1\right] I_1 \\
&\quad +\Lambda(\rho_2+\mu+\mu_2)\left[\frac{2\tau+(\mu S-\Lambda)(\frac{\rho_2}{\Lambda})}{\rho_2+\mu+\mu_2} - 1\right] I_2 \\
&\quad +\Lambda(\mu+\mu_1+\mu_2)\left[\frac{2(\beta+\tau)}{\mu+\mu_1+\mu_2} - 1\right] I_3 \\
&< 0,
\end{aligned}
$$

by using the hypothesis (B.7). Therefore, $dL/dt < 0$ everywhere except at E_0^*, where $dL/dt = 0$. This implies that the disease-free equilibrium E_0^* is globally asymptotically stable. □

In Proposition B2, $\bar{S}$ represents an upper limit on the size of the susceptible population, S.

B.2.2. Quasi-Disease-Free Equilibria (QDFE)

First QDFE. The first quasi-disease-free equilibrium for the system given by (B.1-B.4) is:

$$
E_1^* = \left(\frac{\Lambda}{\mu\Re_0^{1*} + \mu_1(\Re_0^{1*}-1)}, \frac{\Lambda(\Re_0^{1*}-1)}{\mu\Re_0^{1*} + \mu_1(\Re_0^{1*}-1)}, 0, 0\right). \tag{B.8}
$$

E_1^* exists if $\Re_0^{1*} > 1$.

PROPOSITION B3. *(Local Stability of First QDFE in SI × SI Model with Recovery)* *The first QDFE E_1^*, which exists if $\Re_0^{1*} > 1$, is locally asymptotically stable provided that*

$$
\begin{aligned}
\rho_1+\rho_2+\mu+\mu_1+\mu_2 > \tau\left[1+\frac{\mu_1}{\Re_0^{1*}(\beta-\rho_1+\rho_2-\mu_1+\mu_2)}\right] \\
+\frac{\rho_1(\tau+\beta-\rho_1-\mu-\mu_1)}{\beta-\rho_1+\rho_2-\mu_1+\mu_2}.
\end{aligned} \tag{B.9}
$$

Proof of Proposition B3. *(Local Stability of First QDFE)*

The proof of Proposition B3 is similar to Proposition 3. □

Second QDFE. The second quasi-disease-free equilibrium for the system given by (B.1-B.4) is:

$$E_2^* = \left(\frac{\Lambda}{\mu \Re_0^{2*} + \mu_2(\Re_0^{2*} - 1)}, 0, \frac{\Lambda(\Re_0^{2*} - 1)}{\mu \Re_0^{2*} + \mu_2(\Re_0^{2*} - 1)}, 0 \right). \tag{B.10}$$

E_2^* exists if $\Re_0^{2*} > 1$.

Proposition B4. *(Local stability of Second QDFE in SI × SI Model with Recovery)* *The second QDFE E_2^*, which exists if $\Re_0^{2*} > 1$, is locally asymptotically stable provided that*

$$\begin{aligned} \rho_1 + \rho_2 + \mu + \mu_1 + \mu_2 > \beta \left[1 + \frac{\mu_2}{\Re_0^{2*}(\tau + \rho_1 - \rho_2 + \mu_1 - \mu_2)} \right] \\ + \frac{\rho_2(\tau + \beta - \rho_2 - \mu - \mu_2)}{\tau + \rho_1 - \rho_2 + \mu_1 - \mu_2}. \end{aligned} \tag{B.11}$$

Proof of Proposition B4. *(Local Stability of Second QDFE)*

The proof of Proposition B4 is similar to Proposition B3, because of symmetry. □